## Indiana State Mathematics Contest 2016

## Comprehensive

# Do not open this test booklet until you have been advised to do so by the test proctor. 

This test was prepared by faculty at Ball State University

Next year's math contest date: Saturday, April 22, 2017

1. The figure below consists of a square ABCD and a half-circle CPD . The point P is the midpoint of the arc CPD and the point Q is the midpoint of the side BC . If the length of side AB is 10 , what is the area of the shaded region?

(A) $\frac{25}{2}+\frac{125}{12} \pi$
(B) $\frac{25}{2}+\frac{25}{2} \pi$
(C) $\frac{45}{3}+\frac{25}{2} \pi$
(D) $\frac{50}{3}+\frac{25}{2} \pi$
(E) $\frac{275}{12}+\frac{25}{2} \pi$
2. Draw a vertical line of length 1 . At the end of this line draw 2 lines that are half the length of the previous line and at 45 degree angles to it. Repeat this procedure with each line generated. Below is a picture of the result after this procedure was repeated 4 times after drawing the original vertical line.


The limit of the vertical height of this tree, as this procedure is repeated an infinite number of times, equals $\frac{a}{b}\left(1+\frac{1}{2 \sqrt{2}}\right)$, where $a$ and $b$ have no common factors except 1 . What is the value of $a+b$ ?
(A) 4
(B) 5
(C) 6
(D) 7
(E) 8
3. There is a monkey and a bag that contains 2016 peanuts. On the first day, the monkey will take 1 peanut from the bag. On each day following the first, the monkey will take a number of peanuts from the bag equal to the sum of the peanuts that were taken on all previous days.

If there exists a day where the number of peanuts remaining in the bag is less than the total number of peanuts already taken, then the monkey will again take 1 peanut from the bag and repeat the same action as before.

How many days will it take for the monkey to take all of the peanuts from the bag?
(A) 50
(B) 51
(C) 52
(D) 53
(E) 54
4. $0.1 \times 0.3+0.2 \times 0.4+0.3 \times 0.5+0.4 \times 0.6+\cdots+9.7 \times 9.9+9.8 \times 10.0=$ ?
(A) 3088.48
(B) 3184.51
(C) 3282.51
(D) 3382.50
(E) 3484.50
5. Begin with the 50 numbers $\{1,2,3,4, \ldots, 50\}$. Randomly place these 50 numbers into 10 groups of 5 numbers each. Calculate the median of each of these 10 groups. What is the maximum value of the sum of these 10 medians?
(A) 342
(B) 343
(C) 344
(D) 345
(E) 346
6. If $x+2 y-5 z=3$ and $x-2 y-z=-5$, where $x, y$, and $z$ are all Real numbers, then what is the minimum value of $x^{2}+y^{2}+z^{2}$ ?
(A) $\frac{23}{5}$
(B) $\frac{54}{11}$
(C) $\frac{50}{9}$
(D) 5
(E) 53
7. If $x$ and $y$ satisfy the following inequalities: $\quad x-y \geq-1 ; 2 x+y \geq 3 ; x \leq 2 ; \quad y \geq 1$ then what is the maximum value of $3 x-6 y$ ?
(A) -20
(B) -12
(C) -3
(D) 0
(E) 8
8. Suppose that $a$ is a positive integer. After solving $\frac{a x+1}{x^{2}+2 a x+1} \geq 1$, we see that the solution set is $(-3-$ $2 \sqrt{2},-3] \cup(-3+2 \sqrt{2}, 0]$. What is the value for $a$ ?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
9. If $x, y, z \in[0,1]$, then what is the maximum value of $\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}$ ?
(A) $\frac{3 \sqrt{2}}{2}$
(B) $\frac{5}{2}$
(C) $\sqrt{2}+1$
(D) $3 \sqrt{2}-2$
(E) $\frac{8 \sqrt{2}}{3}$
10. Suppose that, in radians, the angles $A$ and $B$ are given by $A=\frac{\pi}{2}$ and $B=\tan ^{-1}\left(-\frac{4}{3}\right)$. Calculate

$$
\tan (A+B)
$$

(A) $\frac{13}{10}$
(B) $\frac{133}{100}$
(C) $-\frac{3}{5}$
(D) $-\frac{4}{5}$
(E) $-\frac{4}{3}$
11. There are 4 different passwords A, B, C, D. Each week a new password is randomly chosen from the 3 passwords that were not used the previous week.

If password A is chosen for the 1 st week, what is the probability that password A is chosen for the 7 th week?
(A) $\frac{45}{243}$
(B) $\frac{49}{243}$
(C) $\frac{52}{243}$
(D) $\frac{61}{243}$
(E) $\frac{65}{243}$
12. Consider quadrilateral ABCD , where $\angle \mathrm{B}=135^{\circ}, \angle \mathrm{C}=120^{\circ}, \mathrm{AB}=2 \sqrt{3}, \mathrm{BC}=4-2 \sqrt{2}$, and $\mathrm{CD}=4 \sqrt{2}$. What is the length of AD ?

(A) $2+2 \sqrt{6}$
(B) 7
(C) $3 \sqrt{6}$
(D) $3+2 \sqrt{6}$
(E) $4+2 \sqrt{6}$
13. You are given the 5 points $A=(1,1), B=(2,-1), C=(-2,-1), D=(0,0)$, and $P_{0}=(0,2)$. $\mathrm{P}_{1}$ equals the rotation of $\mathrm{P}_{0}$ around A by $180^{\circ}, \mathrm{P}_{2}$ equals the rotation of $\mathrm{P}_{1}$ around B by $180^{\circ}$, $P_{3}$ equals the rotation of $P_{2}$ around C by $180^{\circ}, \mathrm{P}_{4}$ equals the rotation of $\mathrm{P}_{3}$ around D by $180^{\circ}$, $\mathrm{P}_{5}$ equals the rotation of $\mathrm{P}_{4}$ around A by $180^{\circ}$, and so on repeating this pattern.

If $\mathrm{P}_{2016}=(a, b)$, then what is the value of $a+b$ ?
(A) 2018
(B) 4030
(C) 4036
(D) 8066
(E) 16126
14. There are 6 cards, each with a number written on it. The 6 numbers are $2,2,4,4,6,6$ respectively. If you randomly choose 3 cards, what is the probability that the 3 numbers on the cards can represent the lengths of 3 sides of a triangle that has non-zero area?
(A) $\frac{2}{5}$
(B) $\frac{1}{2}$
(C) $\frac{3}{5}$
(D) $\frac{2}{3}$
(E) $\frac{3}{4}$
15. Let $x, y, z$ be non-negative real numbers such that $x+y+z=1$. What is the maximum value of $2 x y+y z+2 z x$ ?
(A) $\frac{12}{25}$
(B) $\frac{1}{2}$
(C) $\frac{9}{16}$
(D) $\frac{5}{9}$
(E) $\frac{4}{7}$
16. In the casino game of Craps, the shooter rolls two standard 6 -sided dice. If the sum of the two dice is a 4,5 , $6,8,9$, or 10 then this number becomes the shooter's "Point". The shooter will then continue to roll the dice again and again until either the sum of the two dice is a 7, in which case the shooter loses, or the sum of the two dice equals the original "Point", in which case the shooter wins.

For example, the sequence of rolls $5,2,8,10,5$ has 5 as the "Point" and results in a win for the shooter. However, the sequence of rolls $9,5,7$ has 9 as the "Point" but results in a loss for the shooter.

A shooter rolls the two dice and the result is a sum of 8 , which becomes the "Point".
What is the probability that the shooter wins? In other words, what is the probability that another 8 is rolled before a 7 is rolled?
(A) $\frac{25}{396}$
(B) $\frac{5}{36}$
(C) $\frac{1}{3}$
(D) $\frac{4}{9}$
(E) $\frac{5}{11}$
17. An investment analyst has tracked a certain stock for the past 6 months and has found that on any given day it either goes up 1 point or down 1 point. Furthermore, it went up 1 point on $25 \%$ of the days and down 1 point on $75 \%$. What is the probability that at the end of the day 4 days from now, the price of the stock is the same as it is today? Assume that the daily fluctuations are independent events.
(A) 0.035
(B) 0.141
(C) 0.211
(D) 0.246
(E) 0.281
18. In the grid below each row, column, and $3 \times 3$ box must contain exactly one of each digit from 1 to 9 . What is the value of the digit X ?

| 7 | 2 |  |  | 6 |  |  | 3 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | $\mathbf{X}$ |  |  | 2 |  |  |  | 5 |
|  |  | 5 |  |  |  | 2 |  |  |
|  |  |  | 6 |  | 7 |  |  |  |
| 8 | 5 |  |  |  |  |  | 1 | 7 |
|  |  |  | 5 |  | 4 |  |  |  |
|  |  | 2 |  |  |  | 7 |  |  |
| 9 |  |  |  | 7 |  |  |  | 2 |
| 1 | 7 |  |  | 5 |  |  | 6 | 8 |

(A) 1
(B) 4
(C) 6
(D) 8
(E) 9
19. If $\lim _{n \rightarrow \infty}\left(\left[\begin{array}{cc}1 & 5 / 6 \\ 0 & 1 / \sqrt{3}\end{array}\right]^{n}\right)=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$, then $b=\frac{x}{y}+\frac{5 \sqrt{3}}{12}$ where $x$ and $y$ have no common factors except 1 .

Then what is the value of $x+y$ ?
(A) 7
(B) 8
(C) 9
(D) 10
(E) 11
20. It is known that each of 4 people $A, B, C, D$ tells the truth 1 out of 3 times and lies 2 out of 3 times.

Suppose that A makes a statement, and then D says that C says that B says that A was telling the truth. What is the probability that A was actually telling the truth?
(A) $\frac{3}{10}$
(B) $\frac{13}{41}$
(C) $\frac{1}{3}$
(D) $\frac{11}{32}$
(E) $\frac{41}{81}$
21. On an exam, the average score of those passing is $85 \%$ and the average score of those failing is $55 \%$. If the average score of everyone who took the exam is $73 \%$, what percent of the students passed?
(A) $58 \%$
(B) $59 \%$
(C) $60 \%$
(D) $61 \%$
(E) $62 \%$
22. Two trains are moving at constant speed along tracks that are parallel to each other. One train is 350 feet long and the other is 450 feet long. When moving in opposite directions, the trains can pass each other in 8 seconds. When moving in the same direction, the faster train completely passes the slower train in 16 seconds. Let $S$ be the speed of the slower train and $F$ be the speed of the faster train.

What is the value of $3 \mathrm{~S}+2 \mathrm{~F}$ ?
(A) $225 \mathrm{ft} / \mathrm{sec}$
(B) $245 \mathrm{ft} / \mathrm{sec}$
(C) $275 \mathrm{ft} / \mathrm{sec}$
(D) $285 \mathrm{ft} / \mathrm{sec}$
(E) $300 \mathrm{ft} / \mathrm{sec}$
23. Grass in a farmer's pasture grows at a constant and uniform rate. The grass can support 16 big cows for 20 days or 80 small cows for 10 days. If a big cow eats 3 times more grass than a small cow, how many days can the grass in this pasture support 12 big cows and 60 small cows eating together?
(A) 6
(B) 8
(C) 10
(D) 12
(E) 14
24. A group of men working together at the same rate can normally finish a job in 45 hours. However, in this problem the men report to work one at a time, arriving at equally-spaced intervals, over a period of time. Once on the job each man will stay and work until the job is finished.

If the first man works 5 times as many hours as the last man, how many hours does the first man work?
(A) 55
(B) 60
(C) 65
(D) 70
(E) 75
25. If $\sum_{n=2}^{50} \frac{1}{\log _{n}(e)}=\frac{1}{\log _{2}(e)}+\frac{1}{\log _{3}(e)}+\frac{1}{\log _{4}(e)}+\mathrm{L}+\frac{1}{\log _{50}(e)}=\ln (a)$, then what is the value of $a$ ?
(A) $50 \times 50$
(B) 50 !
(C) $50^{50}$
(D) $2+3+4+\mathrm{L}+50$
(E) $2^{50}$
26. An airplane flying from New York to London has 9 boys, 5 American children, 9 men, 7 British boys, 14 Americans, 6 American males, and 7 British females. How many people are on the airplane?
(A) 29
(B) 31
(C) 33
(D) 35
(E) 37
27. If $\binom{n}{r}=\frac{n!}{r!(n-r)!}$ and $k!=k \times(k-1) \times(k-2) \times(k-3) \times \mathrm{L} \times 3 \times 2 \times 1$,
then what is the largest 3-digit prime factor of $\binom{2000}{1000}$ ? Below is a table of all 3-digit prime numbers:

| 101 | 103 | 107 | 109 | 113 | 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 | 179 | 181 | 191 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 193 | 197 | 199 | 211 | 223 | 227 | 229 | 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 | 283 |
| 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 | 353 | 359 | 367 | 373 | 379 | 383 | 389 | 397 | 401 |
| 409 | 419 | 421 | 431 | 433 | 439 | 443 | 449 | 457 | 461 | 463 | 467 | 479 | 487 | 491 | 499 | 503 | 509 |
| 521 | 523 | 541 | 547 | 557 | 563 | 569 | 571 | 577 | 587 | 593 | 599 | 601 | 607 | 613 | 617 | 619 | 631 |
| 641 | 643 | 647 | 653 | 659 | 661 | 673 | 677 | 683 | 691 | 701 | 709 | 719 | 727 | 733 | 739 | 743 | 751 |
| 757 | 761 | 769 | 773 | 787 | 797 | 809 | 811 | 821 | 823 | 827 | 829 | 839 | 853 | 857 | 859 | 863 | 877 |
| 881 | 883 | 887 | 907 | 911 | 919 | 929 | 937 | 941 | 947 | 953 | 967 | 971 | 977 | 983 | 991 | 997 |  |

(A) 631
(B) 653
(C) 661
(D) 673
(E) 709
28. How many integers greater than 9 have the property that each digit, except the first, is strictly greater than the digit before it? One example of an integer with this property is 3569 . Two examples of integers without this property are 3549 and 3559 .
(A) 256
(B) 382
(C) 466
(D) 502
(E) 511

